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ON AN OIL-FIELD CONTOUR†

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The problem of the displacement of the interface between water and oil of an initial form such that a cusp is formed on this interface at a certain instant of time is considered in greater detail than in previous investigations. It is assumed that the viscosity of the water is many time smaller than that of the oil. Problems concerning the displacement of similar lines (under more general assumptions) are briefly reviewed. © 1997 Elsevier Science Ltd. All rights reserved.

We will consider the simplest problem. Suppose that a closed contour $\mathcal{L}$ (Fig. 1), surrounded by water, lies in the horizontal (x, y) plane and that, at a point A within this contour, there is point sink which approximates a well from which oil is pumped. In practice when the difference between the viscosities of the water and the oil is taken into account, a breakthrough of water into the well occurs at a certain instant of time along the line BA which corresponds to the shortest distance from the oil-field contour to the well. However, if it is assumed that the viscosity of the external fluid (water) is equal to zero, the pressure p at the contour $\mathcal{L}$ will remain constant $p = p_0$ all the time, from which it follows that the time-dependent line on which the velocity potential $\varphi(x, y, t)$ is constant will be the contour of the domain [1, 2]

$$\varphi(x, y, t) = -\varkappa_0 p_0/\mu = \text{const} \tag{1}$$

Here μ is the oil dynamic coefficient of viscosity and $\varkappa_0$ is the soil permeability.

Differentiating Eq. (1) with respect to t, we obtain a non-linear boundary condition, which must be satisfied on the as yet unknown line

$$\sigma\partial\varphi/\partial t + (\partial\varphi/\partial x)^2 + (\partial\varphi/\partial y)^2 = 0 \tag{2}$$

Here σ is the soil porosity

Hence, in this a simplified scheme [1–3], the problem of the flow of oil surrounded by water in a headed reservoir reduces to finding the harmonic function $\varphi(x, y, t)$ with condition (2) on the contour $\mathcal{L}$, the initial shape of which is specified.

The solution of this problem using the Dupuit formula for the flow rate in the case of the inflow of oil to an ideal well is sought [2, 3] in the form of the series

$$z = A_1(t)\zeta + A_2(T)\zeta^2 + \ldots + A_n(t)\zeta^n + \ldots, \quad \zeta = e^{i\theta}, \quad z = x + iy \tag{3}$$

where the complex variable $\zeta = \xi + i\eta$ is such that the point $\zeta = \zeta_0$ corresponds to the well centre at the point A and the circle $\zeta = e^{i\theta}$, where $0 \leqslant \theta \leqslant 2\pi$, corresponds to the contour $\mathcal{L}$. The functions $A_i(t)$ $(i = 1, 2, \ldots)$ are real-valued functions of time t.

Solutions of the problem of the form (3) obtained by Kochina [2, 3] have also been described in detail in [4].

The case when the contour $\mathcal{L}$ is initially a circle of unit radius with an eccentrically located well and $\zeta_0 = c$ is a real number was considered in [2]. The complete solution of the problem can be found. Series (3) consists of an infinite number of terms. Good agreement was obtained with the results of experiments [5].

A second example was considered in [3]. The equation of the initial oil-field contour was given in the parametric form

$$z = \zeta + a\zeta^2 \tag{4}$$

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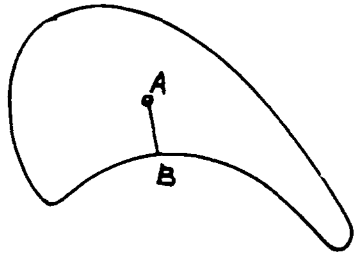

Fig. 1.

$$\zeta = e^{i\theta} \tag{5}$$

where a is a real, positive parameter and $\zeta = 0$, that is, the well A is located at the point $z = \zeta = 0$.

It can be shown that, in this case, series (3) is a second-degree polynomial

$$z = A_1(t)\zeta + A_2(t)\zeta^2 \tag{6}$$

Solution of the form

$$z = A_1(t)\zeta + A_2(t)\zeta^2 + \ldots + A_n(t)\zeta^n \tag{7}$$

were subsequently called polynomial plane algebraic curves.

We shall now consider the initial curves (4) and (5) for various values of the parameter a in greater detail.

When $a = 0$, we have a circle with the well at the centre.

Equation (4) can be rewritten in the explicit form

$$x = \cos\theta + a\cos 2\theta, \quad y = \sin\theta + a\sin 2\theta \tag{8}$$

It has been shown in [3] that the curve (4), (5), (or (8), (5)) is a cardioid when $a = 1/2$.

Putting $x = x' - a$, $y = y'$ in formulae (8), we obtain the equation of the limaçon of Pascal in polar coordinates

$$r = k\cos\varphi + l \tag{9}$$

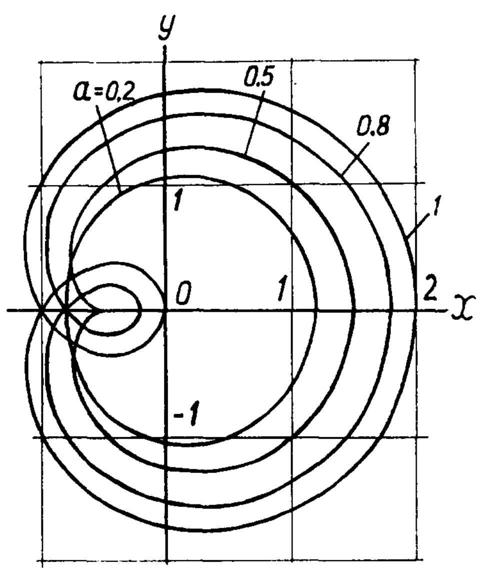

Fig. 2.

where $k = 2a, l = 1$, in the coordinates (x', y'). When $a = 1/2$, this is a cardiod, a curve with a corner point.

It is convenient to distinguish five cases: (1) $0 < a < 0.25$; (2) $a = 0.25$; (3) $0.25 < a < 0.5$; (4) $a = 0.5$; (5) $a > 0.5$.

We shall dwell on the fifth case when there is a loop. It follows from (8) that $y = 0$ when $\theta = 0, \pi$ and $\cos\theta_0 = -1/2a$ where $a > 1/2$ and $x = \cos\theta_0 + a\cos 2\theta_0 = -a$. This is the start of the loop. When $\theta = \pi$, we have $x = a - 1$ and this is the end of the loop. Four curves (in x, y coordinates in accordance with formulae (8)) are shown in Fig. 2: for $a = 0.2$ (the first case), 0.5 (the fourth case), and 0.8 and 1 (the fifth case). In the Cartesian coordinates x, y, these are plane algebraic curves of the tenth order (of the eighth order when $a = 1$) which have an isolated singular point $x = y = 0$ (there is no such point when $a = 1$) in addition to the singular points which exist in the case of the limaçon of Pascal and a cardioid.

The solution of the problem of the motion of an oil-field contour, that is, the interface between the water and the oil, subject to the condition that the initial contour has the form (4), (5) has been given, as already noted above, in [3] and it has also been described in [4] and has the form (6), where $A_i(t)$ ($i = 1, 2$) are certain functions which are determined during the course of the solution and $z/A_1(t)$ is "as though of the form" (4) with a parameter a which increases with time.

The results of this solution are: if $a \leqslant 1/2$ then a cusp $x = -(a/4)^{1/3}, y = 0$ is formed in the contour at a certain instant of time $t_1 \geqslant 0$, and, when $a = 1/2$, we have $x = -1/2, y = 0$ (there is already a cusp $x = -1/2, y = 0$ in the initial contour in this case). When $a > 1/2$, there is a loop in the contour (8), (5), and this contour cannot be taken as the initial contour.

The solution of the problem therefore ceases to be one-sheeted until the instant the water breaks through into the well (starting from the instant of time t_1).

Many investigators have dealt with problems of the motion of an oil-field contour. We will now consider some of the results obtained.

Galin [6] showed that, under polynomial instant conditions, there is a loss of one-sheetedness of the solution at a certain instant of time prior to the instant when water breaks through into the well, and this result is completely in agreement with the examples considered above [2, 3]. This phenomenon does not occur in the case of an initial contour in the form of a circle (unlike the case of an initial contour of the form of (4), (5)).

The case when fluid is forced into the well has also been considered in a number of papers. In this case, the domain of motion does not contract with time but expand, and no corner point is formed. In the Dupuit formula, it is now necessary to assume that the flow rate is positive rather than negative.

The third-order polynomial mappings

$$z = A_1(t)\xi + A_2(t)\xi^2 + A_3(t)\xi^3$$

have been considered in [7].

Particular attention was paid to the formation of corner points in such motions. In the general case, the singularity is the same as in the problem with a second-order polynomial which has been described above, but, in the case of specially adjusted initial data, a singularity of order $^5/_2$ is obtained instead of this singularity of order 3/2.

Loss of one-sheetedness of the solution also occurs before the instant when water first enters the well in problems of this kind with the oil withdrawal from the domain.

Mathematically similar problems, with the formation of cusps on the periphery, have been considered in [8], where Richardson's method [9, 10] is followed; the solutions of the problems are also not continuable after the formation of cusps. Closely related problems are considered in [11–14].

Views on the failure to take account of capillary forces in the problem under consideration and on the fact that it is impossible to discard inertial terms in the equations of motion while retaining the quadratic terms in condition (2) have been expressed by Danilov [15]. He showed that, if account is taken of the action of surface tension forces, then a cusp does not occur at the interface of immiscible fluids. However, the calculation was only carried out for the case of an initial contour in the form of a circle with an eccentrically located well.

Hitherto, it has been assumed that the water viscosity μ_* can be neglected compared with the oil viscosity μ. We shall now drop this assumption.

A series of experiments, which represented a model of the contraction of an "oil-field contour" [5] in a horizontal Hele–Shaw cell, have been carried out at the Institute of Mechanics of the Academy of Sciences of the USSR. Similar experiments have been conducted under Danilov's guidance [15]. A fairly accurate form of the subsequent contours was obtained in these experiments for an initial circular shape of the contour which has been considered theoretically above. In the domain which is closest to the "well", a contour with smooth outlines, without the formation of a cusp up to arrival at the well itself, was always obtained. However, there can be sharp changes in the shape of the contour at high local velocities, and the penetration of the less dense fluid into the denser fluid is unstable.

Note that the problem of the stability of its motion with respect to infinitely small perturbations is intimately associated with the problem of the progress of such a contour. Such problems have been investigated in [16–18].

Displacement instability affects the development of deposits of highly viscous oil under waterdrive conditions. An experimental investigation of such instability during the displacement of oil by water has been carried out on transparent horizontal models of an unconsolidated rock reservoir [19]. It was shown that, if $\mu/\mu_* \leqslant 10$–13, then, even at comparatively high velocities, flow under the action of capillary forces becomes stable. However, if $\mu/\mu_* > 13$, then, for the flow to be stable, it is necessary to reduce the velocity down to very small values.

A group of foreign scientists [20] has made a theoretical investigation of the effect of capillary forces on the stability of the interface. It was found that, at sufficiently low velocities, the front motion may become stable due to capillary forces even for an unfavourable ratio of the viscosities. However, in order to achieve quantitative agreement with experiment, the concept of an "effective" surface tension had to be introduced, which was about 40 times greater than the true surface tension. This discrepancy was caused by the fact that it was assumed that the interface between the phases was infinitely thin and smooths itself out because of the capillary pressure P. Such smoothing does, in fact, occur due to the dissipative action of the actual capillary pressure in a porous medium P_1, and the ratio of the quantities P/P_1 of the order of r/R, where r is the curvature of the menisci and R is the curvature of the interface.

In the general case, assuming that the reservoir is horizontal and located between two impermeable reservoirs, that there is a single perfect well within the oil-bearing contour and $\mu_* \neq 0$, the motion of both the water and the oil will occur in doubly-connected domains [21, 22].

We shall assume that $\mu = \mu_*$ and that the oil-field contour is initially a circle of radius a, the contour of the reservoir charging domain is also a circle but of significantly greater radius on which the pressure is constant ($p = p_0$) and at the centre of which there is a well. Here, the well is either located at the centre of the oil-field contour or at some other point within this contour, and the well has the form of a circle of small radius r_A on which the pressure is constant ($p = p_1 < p_0$).

In such a case, the complex potential of the steady-state motion in the well has the form

$$w = -\frac{Q}{2\pi}\ln z$$

where Q is the production rate of the well per unit capacity of the reservoir. The dependence of the radius of the "oil-field contour" on the time t takes the form [21–23]

$$r = \sqrt{r_0^2 - Qt/(\pi\sigma)} \tag{10}$$

Here $r_0 = r_0(\theta)$ is the radius vector of the initial oil-field contour and θ is the polar angle.

It is clear from formula (10) that, at the instant of time

$$t_A = \pi\sigma(r_0^2 - r_A^2)/Q \tag{11}$$

all the oil will be pumped off by the well and, when $t > t_A$, only water will enter the well if the well is located at the centre of the initial oil-field contour ($r_0 = a$). Otherwise, a mixture of oil and water will enter the well when $t \geqslant t_A$ and the shape of the contour will change with time, depending on θ and t. It is necessary to replace r_0 by $r_{0\min}$ in formula (11). At a certain instant of time $t_1 > t_A$, it becomes uneconomic to draw off the oil together with the water and the well will be shut down with the residual oil remaining in the reservoir.

If the well is located on the circle itself and $r_A = 0$, then $t = 4\pi\sigma a^2\cos^2\theta/Q$, and water already starts to enter the well at the initial instant of time $t = 0$, when $\theta = \pi/2$, and all the oil will be withdrawn by the well at the instant $t_A = 4\pi\sigma a^{2/Q}$.

In this case, the equation of curve (10) in Cartesian coordinates can be written in the form ($\tau = Qt/(\pi\sigma)$)

$$(x^2 + y^2 + \tau)y^2 = x^2(4a^2 - x^2 - y^2 - \tau) \tag{12}$$

This is a plane algebraic curve of the fourth order; $x = y = 0$ is a nodal point.

Curves (12) (where τ is parameter) are a special case of Perseus curves, that is, the lines of intersection of the surface of a torus by a plane which is parallel to the axis of the torus and is tangential to an internal part of its surface (when $\tau = 2a^2$, curve (12) is Bernoulli's lemniscate).

On changing to Cartesian coordinates (x, y) in formula (x, y) in formula (10), we obtain sixth-order plane algebraic curves (up to the instant when water enters the well, the centre of the well (the point $x = y = 0$) is an isolated singular point and a nodal point after that instant).

In general, problems on the co-current filtration flow of two or more fluids with different physical properties, including problems on an oil-field contour without the simplifying assumptions which have been considered here, are very complex. A number of such plane and spatial problems has been described in [21, 24].

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